# Long Time Asymptotics of the Parabolic Anderson Model in the Hyperbolic Space

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# The PAM in the Euclidean Space

**The parabolic Anderson model** (PAM) on $\mathbb{R}^d$:

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \xi(x) \cdot u(t,x), \\ u(0,\cdot) \equiv 1. \end{cases}$$

- Here $\xi = \{\xi(x) : x \in \mathbb{R}^d\}$ is a mean-zero, stationary Gaussian field with sufficiently regular sample functions.
- The covariance function

$$Q(x) \triangleq \mathbb{E}[\xi(0)\xi(x)]$$

is twice continuously differentiable at the origin.

# The PAM in the Euclidean Space

**Feynman-Kac representation**:

$$u(t,x) = \mathbb{E}_x\left[e^{\int_0^t \xi(W_s)ds}\right],$$

where $W$ is an BM on $\mathbb{R}^d$ independent of $\xi$.

**Basic questions**:

- What is the exact long time growth of $u(t,x)$ as $t \to \infty$?
- What is the fluctuation asymptotics?
- Moment vs almost sure asymptoics.

# The PAM in the Euclidean Space

Define

$$\sigma^2 \triangleq \mathrm{Var}[\xi(0)], \ \chi = \frac{1}{\sqrt{2}} \mathrm{Tr}\sqrt{-\mathrm{Hess}Q(0)}.$$

- **Moment asymptotics** (Gärtner-König, AAP 2000):

$$\langle u(t,x)^p \rangle = \exp\left( \frac{\sigma^2 p^2}{2} t^2 - \chi p^{3/2} t^{3/2} (1 + o(1)) \right), \quad \forall p \geqslant 1.$$

- **Almost sure asymptotics** (Gärtner-König-Molchanov, PTRF 2000):

$$u(t,x) = \exp\Big( \sqrt{2d\sigma^2} t (\log t)^{1/2} \\ -(2d/\sigma^2)^{1/4} \chi t (\log t)^{1/4} (1 + o(1)) \Big).$$

# Basic questions

If one changes the underlying geometry, would it lead to different asymptotic behaviours of the PAM?

Some aspects of the geometry of non-negatively curved spaces are "not-so-much" different from $\mathbb{R}^d$:

- Volume growth, heat kernel decay, behaviour of BM etc.

We work with the geometry of negative curvature:

- The standard hyperbolic space $\mathbb{H}^d$.

# Some related works in non-Euclidean spaces

- Compact Riemannian manifolds (Chen-Ouyang-Vickery, 2023).
- Trees (Hollander-König-Santos, 2020; Hollander-Wang, 2023).
- Heisenberg groups (Baudoin-Ouyang-Tindel-Wang, 2023)
- Metric measure spaces (Baudoin-Chen-Huang-Ouyang-Tindel-Wang, 2024).
- Singular case on compact Rumanian surfaces (Dahlqvist-Diehl-Driver, 2017).

# The hyperbolic plane

**The upper-half plane**: $\mathbb{H}^2 \triangleq \{(x, y) : y > 0\}$.

Metric tensor:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

- Complete, simply-connected Riemannian manifold of curvature $-1$.

Geodesics:

- Semi-circles with both ends perpendicular to the $x$-axis.

Hyperbolic distance:

$$d(z_1, z_2) = 2\text{arcsinh}\frac{|z_1 - z_2|_{\text{eu}}}{2\sqrt{y_1 y_2}}, \quad z_i = (x_i, y_i).$$

# The hyperbolic space

Hyperbolic Laplacian:

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Volume form:

$$\mathrm{vol} = \frac{dxdy}{y^2}.$$

Group of orientation-preserving isometries:

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\},$$

where

$$\mathrm{SL}_2(\mathbb{R}) = \{A \in \mathrm{Mat}_2(\mathbb{R}) : \det A = 1\}$$

acts on $\mathbb{H}^2$ via Möbius transform.

# Some global properties

**Exponential volume growth**:

$$\mathrm{vol}(B_R) = 2\pi\,(\cosh R - 1) \propto e^R.$$

**Bottom spectrum of Laplacian**:

$$\inf \mathrm{Spec}(-\Delta) = \frac{1}{4}.$$

**Exponential decay of heat kernel** (Davies-Mandouvalos, 1998):

$$p(t,x,y) \asymp \frac{1}{t} \exp\left( -\frac{1}{4}t - \frac{d(x,y)^2}{4t} - \frac{1}{2} d(x,y) \right) \frac{1 + d(x,y)}{\sqrt{1 + d(x,y) + t}}$$

# Long time behaviour of BM

The *hyperbolic Brownian motion* $W_t$: Markov process generated by $\Delta$.

- Transience:

$$\frac{d(o, W_t)}{t} \to 1 \text{ a.s. } (d(o, W_t) \to \infty).$$

- Central limit theorem (Babillot, 1994):

$$\frac{d(o, W_t) - t}{\sqrt{2t}} \xrightarrow{d} N(0, 1).$$

# Sullivan's theorem

With probability one, $W_t$ converges to a definite (random) point on the boundary at infinity as $t \to \infty$.

- The angular component of $W_t$ converges a.s.
- On the Poincaré disk, $W_t$ converges a.s. to a point on $S^1$.
- Bounded, non-constant harmonic functions:

$$\mathbb{H}^2 \ni x \mapsto \mathbb{E}_x[\varphi(W_\infty)], \quad \varphi \in C(S^1).$$

# PAM in the hyperbolic space

Let $M = \mathbb{H}^d$ ($d$-dimensional, complete, simply-connected R-manifold with curvature $-1$).

Consider the PAM on $M$:

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \xi(x) \cdot u(t,x), \\ u(0,\cdot) \equiv 1. \end{cases}$$

Gaussian field $\xi$:

- Mean-zero, invariant under isometries, sufficiently regular covariance function.

Feynman-Kac:

$$u(t,x) = \mathbb{E}_x \left[ e^{\int_0^t \xi(W_s) ds} \right],$$

where $W$ is the hyperbolic BM starting from $x$.

# Examples of Gaussian field

Let $\Xi$ be the white noise on the isometry group $G$ w.r.t. the Haar measure $dg$.

- $\{\Xi(A) : A \in \mathcal{B}(G), |A|_G < \infty\}$ is a Gaussian family with mean-zero and covariance structure

$$\mathbb{E}[\Xi(A)\Xi(B)] = |A \cap B|_G.$$

Let $f : M \to \mathbb{R}$ be a smooth function with suitable decay at infinity.

Define

$$\xi(x) \triangleq \int_G f(g^{-1} \cdot x)\Xi(dg), \quad x \in M.$$

Covariance function:

$$Q(x, y) = \int_G f(g \cdot x) f(g \cdot y) dg = Q(d(x, y)).$$

# Moment asymptotics

Define

$$H(t) \triangleq \frac{1}{2}\sigma^2 t^2,\ \beta(t) \triangleq t^{3/2}. \quad (\sigma^2 \triangleq \mathrm{Var}[\xi(x)])$$

**Theorem (Xu-G., 2024+)**

*For any $p \geqslant 1$, one has*

$$\lim_{t\to\infty} \frac{1}{\beta(pt)} \log\left(e^{-H(pt)}\langle u(t,x)^p\rangle\right) = -\chi_{\mathrm{eu}} = -\sqrt{\frac{Q''(0)}{2}}d.$$

# The fluctuation exponent $\chi_{\text{eu}}$

Fix a base point $o \in$ M.

- Geodesic polar coordinates: $x = (\rho, \sigma) \in M$.
  - $\rho = d(x, o) \in [0, \infty)$.
  - $\sigma \in ST_o M$: angular component of $x$ w.t.r. $o$.

Define the functional $J : \mathcal{P}_c(M) \to \mathbb{R}$ by

$$J(\mu) \triangleq -\frac{1}{4}Q''(0) \int_{M\times M} d_{\text{eu}}(z_1, z_2)^2 \mu(dz_1)\mu(dz_2),$$

where

$$d_{\text{eu}}(z_1, z_2) \triangleq \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\langle\sigma_1, \sigma_2\rangle_{T_o M}, \quad z_i = (\rho_i, \sigma_i).$$

# The fluctuation exponent $\chi_{\mathrm{eu}}$

Define the *Donsker-Varadhan* functional:

$$\mathcal{S}_{\mathrm{eu}}(\nu) = \begin{cases} \int_{T_o M} |\nabla \phi|^2 dx, & \nu \ll dx, \frac{d\nu}{dx} = \phi^2 \text{ with } \phi \in H^1(T_o M); \\ +\infty, & \text{otherwise.} \end{cases}$$

- This is essentially the Dirichlet form associated with the Euclidean BM (BM on $T_o M$).

**The fluctuation asymptotics**:

$$\lim_{t\to\infty} \frac{1}{\beta(pt)} \log\left(e^{-H(pt)} \langle u(t,x)^p \rangle\right)$$
$$= -\inf\left\{J(\mu) + \mathcal{S}_{\mathrm{eu}}(\exp_o^{-1}\mu) : \mu \in \mathcal{P}_c(M)\right\}$$
$$=: -\chi_{\mathrm{eu}} = -d\sqrt{Q''(0)/2} \quad \text{(Gärtner-König, 2000)}$$

# Almost sure asymptotics

**Theorem (Wang-Xu-G., 2024+)**

*Suppose additionally that the covariance function is compactly supported. There exist deterministic constants $C, c > 0$ such that with probability one,*

$$\overline{\lim_{t\to\infty}} \frac{1}{t^{5/3}} \log u(t,o) \leqslant C, \quad \underline{\lim_{t\to\infty}} \frac{1}{t^{5/3}} \log u(t,o) \geqslant c.$$

**Euclidean case**:

$$u(t,o) = e^{\sqrt{2d\sigma^2}\cdot t\sqrt{\log t}(1+o(1))}.$$

# Almost sure asymptotics

Growth of sample functions for the Gaussian field $\xi$:

$$\overline{\lim_{R\to\infty}} \frac{\max_{x\in B(o,R)} \xi(x)}{\sqrt{2\sigma^2(d-1)R}} = 1 \text{ a.s.}$$

- The hyperbolic BM $d(o, W_t) \approx (d-1)t + O(\sqrt{t})$.
- On the ball of radius $O(t)$,

$$\max \xi \approx O(\sqrt{t}).$$

- Feynman-Kac $u(t, o) = \mathbb{E}_o[e^{\int_0^t \xi(W_s)ds}]$:

$$u(t, o) \lesssim e^{O(t^{3/2})}.$$

- This intuition is NOT true!

# In working progress

Trying to understand which of the following two scenarios is correct:

1. No exact limit:

$$\varlimsup_{t\to\infty} \frac{1}{t^{5/3}} \log u(t,o) \neq \varliminf_{t\to\infty} \frac{1}{t^{5/3}} \log u(t,o).$$

2. Exact first order asymptotics:

$$\lim_{t\to\infty} \frac{1}{t^{5/3}} \log u(t,o) = C.$$

- Fluctuation asymptotics (we believe: in the scale $t^{4/3}$)?

# Idea of proof: moment asymptotics

**Main philosophy of Gärtner-König**:

- Rescaling argument + Feynman-Kac $\Longrightarrow$

$$e^{-H(t)}u(t,o) = \mathbb{E}_o\left[e^{\int_0^{\beta(t)} \xi_t(W_s^t)ds}\right],$$

  - $\beta(t) = t^{3/2}$.
  - $\xi_t$: suitably rescaled version of $\xi$.
  - $W^t$ is a hyperbolic BM starting at $o$.
  - Both $\xi_t(\cdot)$ and $W^t_\cdot$ are considered defined under curvature $\kappa_t \equiv t^{-1/2}$.

# Idea of proof: moment asymptotics

After taking expectation w.r.t. $\xi$:

$$e^{-H(t)}\langle u(t,o)\rangle = \mathbb{E}_o\left[e^{-\beta(t)J_t(L^t_{\beta(t)})}\right].$$

Here

$$L^t_s(dx) \triangleq \frac{1}{s}\int_0^s \mathbf{1}_{\{W^t_r \in dx\}}dr, \quad s>0;$$

$$J_t(\mu) \triangleq -\frac{1}{\beta(t)}\log\langle e^{\beta(t)(\mu,\xi_t)}\rangle, \quad \mu \in \mathcal{P}_c(M).$$

**Key points**:

- $J_t \longrightarrow J$.
- $\{L^t_{\beta(t)} : t>0\}$ satisfies an LDP with rate function $\mathcal{S}_{\text{eu}}$.

# Idea of proof: moment asymptotics

Varadhan's lemma $\implies$

$$\lim_{t\to\infty} \frac{1}{\beta(t)} \log\left(e^{-H(t)}\langle u(t,o)\rangle\right) = -\inf_{\mu\in\mathcal{P}_c(M)} \{J(\mu) + \mathcal{S}_{\mathrm{eu}}(\mu)\} = -\chi_{\mathrm{eu}}.$$

**Main difficulty**:

- One cannot directly prove such an LDP on the non-compact manifold $M$.
- Curvature ($\kappa_t \equiv t^{-1/2}$) is changing simultaneously as the time scale of BM ($\beta(t) = t^{3/2}$) goes to infinity.

# Idea of proof: Almost sure asymptotics

Why is the effective localisation range is of radius $K(t) = O(t^{4/3})$?

Maximum of $\xi$ on $B(o, K(t))$ is

$$h_{K(t)} \triangleq \sqrt{2\sigma^2(d-1)K(t)} \;(+o(h_{K(t)})).$$

**Localisation**:

- Before exiting $B(o, K(t))$, it takes $\delta(t)$ amount of time to a peak "island" $I$ of $\xi$ and stays there over $[\delta(t), t]$.

# Idea of proof: Almost sure asymptotics

$$
\begin{aligned}
u(t,o) \geqslant & \mathbb{E}_o \Big[ e^{\int_0^{\delta(t)} \xi(W_s)ds} \cdot e^{\int_{\delta(t)}^{t} \xi(W_s)ds} \\
& ; W_{\delta(t)} \in I, W|_{[\delta(t),t]} \subseteq I, W|_{[0,t]} \subseteq B(o,K(t)) \Big] \\
& \gtrsim e^{-C\delta(t)\sqrt{K(t)}} \cdot e^{(t-\delta(t))h_{K(t)}} \cdot e^{-\frac{K(t)^2}{4\delta(t)}}.
\end{aligned}
$$

Choose $K(t)$ as large as possible under the constraint:

$$
(t-\delta(t))h_{K(t)} > \frac{K(t)^2}{4\delta(t)} + C\delta(t)\sqrt{K(t)}.
$$

This leads us to choose $\delta(t) \propto t$ and $K(t) = O(t^{4/3})$, resulting in

$$
u(t,o) \geqslant e^{ct^{5/3}+o(t^{5/3})}.
$$

# The End

Thank you for your attention!